

# A Multiplicative Wavelet-based Model for Simulation of a Random Process

IEVGEN TURCHYN

University of Lausanne, Lausanne, Switzerland

*We consider a random process  $Y(t) = \exp\{X(t)\}$ , where  $X(t)$  is a centered second-order process which correlation function  $R(t, s)$  can be represented as  $\int_{\mathbb{R}} u(t, y)\overline{u(s, y)}dy$ . A multiplicative wavelet-based representation is found for  $Y(t)$ . We propose a model for simulation of the process  $Y(t)$  and find its rates of convergence to the process in the spaces  $C([0, T])$  and  $L_p([0, T])$  for the case when  $X(t)$  is a strictly sub-Gaussian process.*

**Keywords** Sub-Gaussian random processes; Simulation

**Mathematics Subject Classification** Primary 60G12

## 1 Introduction

Simulation of random processes is a wide area nowadays, there exist many methods for simulation of stochastic processes (see e.g. [1, 2]).

But there exists one substantial problem: for most of traditional methods of simulation of random processes it is difficult to measure the quality of approximation of a process by its model in terms of “distance” between paths of the process and the corresponding paths of the model. Therefore models for which such distance can be estimated are quite interesting.

There exists a concept for simulation by such models which is called simulation with given accuracy and reliability. Simulation with given accuracy and reliability is considered, for example, in [3, 4].

Simulation with given accuracy and reliability can be described in the following way. An approximation  $\hat{X}(t)$  of a random process  $X(t)$  is built.

The random process  $\hat{X}(t)$  is called a model of  $X(t)$ . A model depends on certain parameters. The rate of convergence of a model to a process is given by a statement of the following type: if numbers  $\delta$  (accuracy) and  $\varepsilon$  ( $1 - \varepsilon$  is called reliability) are given and the parameters of the model satisfy certain restrictions (for instance, they are not less than certain lower bounds) then

$$P\{\|X - \hat{X}\| > \delta\} \leq \varepsilon. \quad (1)$$

Many such results have been proved for the cases when the norm in (1) is the  $L_p$  norm or the uniform norm. But simulation with given accuracy and reliability has been developed so far almost only for processes which one-dimensional distributions have tails which are not heavier than Gaussian tails (e.g. for sub-Gaussian processes).

We consider a random process  $Y(t) = \exp\{X(t)\}$  and a scaling function  $\phi(x)$  with the corresponding wavelet  $\psi(x)$ , where  $X(t)$  is a centered second-order process such that its correlation function  $R(t, s)$  can be represented as

$$R(t, s) = \int_{\mathbb{R}} u(t, \lambda) \overline{u(s, \lambda)} d\lambda.$$

We prove that

$$Y(t) = \prod_{k \in \mathbb{Z}} \exp\{\xi_{0k} a_{0k}(t)\} \prod_{j=0}^{\infty} \prod_{l \in \mathbb{Z}} \exp\{\eta_{jl} b_{jl}(t)\},$$

where  $\xi_{0k}, \eta_{jl}$  are random variables,  $a_{0k}(t), b_{jl}(t)$  are functions that depend on  $X(t)$  and the wavelet.

We take as a model of  $Y(t)$  the process

$$\hat{Y}(t) = \prod_{k=-(N_0-1)}^{N_0-1} \exp\{\xi_{0k} a_{0k}(t)\} \prod_{j=0}^{N-1} \prod_{l=-(M_j-1)}^{M_j-1} \exp\{\eta_{jl} b_{jl}(t)\}.$$

Let us consider the case when  $X(t)$  is a sub-Gaussian process. Note that the class of processes  $Y(t) = \exp\{X(t)\}$ , where  $X(t)$  is a sub-Gaussian

process, is a rich class which includes many processes which one-dimensional distributions have tails heavier than Gaussian tails, e.g. when  $X(t)$  is a Gaussian process the one-dimensional distributions of  $Y(t)$  are lognormal.

We describe the rate of convergence of  $\hat{Y}(t)$  to a sub-Gaussian process  $Y(t)$  in  $C([0, T])$  in such a way: if  $\varepsilon \in (0; 1)$  and  $\delta > 0$  are given and the parameters  $N_0, N, M_j$  are big enough then

$$\mathbb{P} \left\{ \sup_{t \in [0, T]} |Y(t)/\hat{Y}(t) - 1| > \delta \right\} \leq \varepsilon. \quad (2)$$

A similar statement which characterizes the rate of convergence of  $\hat{Y}(t)$  to  $Y(t)$  in  $L_p([0, T])$  is also proved for the case when (2) is replaced by the inequality

$$\mathbb{P} \left\{ \left( \int_0^T |Y(t) - \hat{Y}(t)|^p dt \right)^{1/p} > \delta \right\} \leq \varepsilon.$$

If the process  $X(t) = \ln Y(t)$  is Gaussian then the model  $\hat{Y}(t)$  can be used for computer simulation of  $Y(t)$ .

One of the merits of our model is its simplicity. Besides, it can be used for simulation of processes which one-dimensional distributions have tails which are heavier than Gaussian tails.

## 2 Auxiliary facts

A random variable  $\xi$  is called *sub-Gaussian* if there exists such a constant  $a \geq 0$  that

$$\mathbb{E} \exp\{\lambda \xi\} \leq \exp\{\lambda^2 a^2 / 2\}$$

for all  $\lambda \in \mathbb{R}$ .

The class of all sub-Gaussian random variables on a standard probability space  $\{\Omega, \mathcal{B}, P\}$  is a Banach space with respect to the norm

$$\tau(\xi) = \inf\{a \geq 0 : \mathbb{E} \exp\{\lambda \xi\} \leq \exp\{\lambda^2 a^2 / 2\}, \lambda \in \mathbb{R}\}.$$

A centered Gaussian random variable and a random variable uniformly distributed on  $[-b, b]$  are examples of sub-Gaussian random variables.

A sub-Gaussian random variable  $\xi$  is called *strictly sub-Gaussian* if

$$\tau(\xi) = (\mathbb{E}\xi^2)^{1/2}.$$

For any sub-Gaussian random variable  $\xi$

$$\mathbb{E} \exp\{\lambda\xi\} \leq \exp\{\lambda^2\tau^2(\xi)/2\}, \quad \lambda \in \mathbb{R}, \quad (3)$$

and

$$\mathbb{E}|\xi|^p \leq 2 \left(\frac{p}{e}\right)^{p/2} (\tau(\xi))^p, \quad p > 0. \quad (4)$$

A family  $\Delta$  of sub-Gaussian random variables is called *strictly sub-Gaussian* if for any finite or countable set  $I$  of random variables  $\xi_i \in \Delta$  and for any  $\lambda_i \in \mathbb{R}$

$$\tau^2 \left( \sum_{i \in I} \lambda_i \xi_i \right) = \mathbb{E} \left( \sum_{i \in I} \lambda_i \xi_i \right)^2.$$

A stochastic process  $X = \{X(t), t \in \mathbf{T}\}$  is called *sub-Gaussian* if all the random variables  $X(t), t \in \mathbf{T}$ , are sub-Gaussian. We call a stochastic process  $X = \{X(t), t \in \mathbf{T}\}$  *strictly sub-Gaussian* if the family  $\{X(t), t \in \mathbf{T}\}$  is strictly sub-Gaussian. Any centered Gaussian process is strictly sub-Gaussian.

Details about sub-Gaussian random variables and processes can be found in [5].

We will use wavelets (see [6] for details) for an expansion of a stochastic process. Namely, we use a scaling function  $\phi(x)$  of an MRA and the corresponding wavelet  $\psi(x)$ . Set

$$\phi_{0k}(x) = \phi(x - k), \quad k \in \mathbb{Z},$$

$$\psi_{jl}(x) = 2^{j/2} \psi(2^j x - l), \quad j, l \in \mathbb{Z}.$$

We require orthonormality of the system  $\{\phi(\cdot - k), k \in \mathbb{Z}\}$ . We denote by  $\hat{f}$  the Fourier transform of a function  $f \in L_2(\mathbb{R})$ .

The following statement is crucial for us.

**Theorem 2.1.** ([7]) Let  $X = \{X(t), t \in \mathbb{R}\}$  be centered random process such that for all  $t \in \mathbb{R}$   $\mathbb{E}|X(t)|^2 < \infty$ . Let  $R(t, s) = \mathbb{E}X(t)\overline{X(s)}$  and there exists such a Borel function  $u(t, \lambda)$ ,  $t \in \mathbb{R}, \lambda \in \mathbb{R}$  that

$$\int_{\mathbb{R}} |u(t, \lambda)|^2 d\lambda < \infty \quad \text{for all } t \in \mathbb{R}$$

and

$$R(t, s) = \int_{\mathbb{R}} u(t, \lambda) \overline{u(s, \lambda)} d\lambda.$$

Let  $\phi(x)$  be a scaling function,  $\psi(x)$  — the corresponding wavelet. Then the process  $X(t)$  can be presented as the following series which converges for any  $t \in \mathbb{R}$  in  $L_2(\Omega)$ :

$$X(t) = \sum_{k \in \mathbb{Z}} \xi_{0k} a_{0k}(t) + \sum_{j=0}^{\infty} \sum_{l \in \mathbb{Z}} \eta_{jl} b_{jl}(t), \quad (5)$$

where

$$a_{0k}(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(t, y) \overline{\hat{\phi}_{0k}(y)} dy = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(t, y) \overline{\hat{\phi}(y)} e^{iyk} dy, \quad (6)$$

$$b_{jl}(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(t, y) \overline{\hat{\psi}_{jl}(y)} dy = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(t, y) 2^{-j/2} \exp\left\{i \frac{y}{2^j} l\right\} \overline{\hat{\psi}\left(\frac{y}{2^j}\right)} dy, \quad (7)$$

$\xi_{0k}, \eta_{jl}$  are centered random variables such that

$$\mathbb{E} \xi_{0k} \overline{\xi_{0l}} = \delta_{kl}, \quad \mathbb{E} \eta_{mk} \overline{\eta_{nl}} = \delta_{mn} \delta_{kl}, \quad \mathbb{E} \xi_{0k} \overline{\eta_{nl}} = 0.$$

**Definition.** Condition RC holds for stochastic process  $X(t)$  if it satisfies the conditions of Theorem 2.1,  $u(t, \cdot) \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$  and inverse Fourier transform  $\tilde{u}_x(t, x)$  of function  $u(t, x)$  with respect to  $x$  is a real function.

**Remark 2.1.** Condition RC guarantees that the coefficients  $a_{0k}(t)$ ,  $b_{jl}(t)$  of expansion (5) are real.

Suppose that  $X(t)$  is a process which satisfies the conditions of Theorem 2.1. Let us consider the following approximation (or model) of  $X(t)$ :

$$\begin{aligned}\hat{X}(t) &= \hat{X}(N_0, N, M_0, \dots, M_{N-1}, t) \\ &= \sum_{k=-(N_0-1)}^{N_0-1} \xi_{0k} a_{0k}(t) + \sum_{j=0}^{N-1} \sum_{l=-(M_j-1)}^{M_j-1} \eta_{jl} b_{jl}(t),\end{aligned}\tag{8}$$

where  $\xi_{0k}, \eta_{jl}, a_{0k}(t), b_{jl}(t)$  are defined in Theorem 2.1.

Approximation of Gaussian and sub-Gaussian processes by model (8) has been studied in [7] and [8].

**Remark 2.2.** *If  $X(t)$  is a Gaussian process then we can take as  $\xi_{0k}, \eta_{jl}$  in (8) independent random variables with distribution  $N(0; 1)$ .*

### 3 A multiplicative representation

We will obtain a multiplicative representation for a wide class of stochastic processes.

**Theorem 3.1.** *Suppose that a random process  $Y(t)$  can be represented as  $Y(t) = \exp\{X(t)\}$ , where the process  $X(t)$  satisfies the conditions of Theorem 2.1. Then the equality*

$$Y(t) = \prod_{k \in \mathbb{Z}} \exp\{\xi_{0k} a_{0k}(t)\} \prod_{j=0}^{\infty} \prod_{l \in \mathbb{Z}} \exp\{\eta_{jl} b_{jl}(t)\}\tag{9}$$

*holds, where product (9) converges in probability for any fixed  $t$  and  $\xi_{0k}, \eta_{jl}, a_{0k}(t), b_{jl}(t)$  are defined in Theorem 2.1.*

The statement of the theorem immediately follows from Theorem 2.1.

**Remark 3.1.** *It was shown in [7] that any centered second-order wide-sense stationary process  $X(t)$  which has the spectral density satisfies the conditions*

of Theorem 2.1. The process  $Y(t) = \exp\{X(t)\}$  can be represented as product (9) and therefore the class of processes which satisfy the conditions of Theorem 3.1 is wide enough.

It is natural to approximate a stochastic process  $Y(t) = \exp\{X(t)\}$  which satisfies the conditions of Theorem 3.1 by the model

$$\begin{aligned} \hat{Y}(t) &= \hat{Y}(N_0, N, M_0, \dots, M_{N-1}, t) \\ &= \prod_{k=-(N_0-1)}^{N_0-1} \exp\{\xi_{0k} a_{0k}(t)\} \prod_{j=0}^{N-1} \prod_{l=-(M_j-1)}^{M_j-1} \exp\{\eta_{jl} b_{jl}(t)\} = \exp\{\hat{X}(t)\}. \end{aligned} \quad (10)$$

**Remark 3.2.** If  $X(t) = \ln Y(t)$  is a Gaussian process then we can use the model  $\hat{Y}(t)$  for computer simulation of  $Y(t)$ , taking as  $\xi_{0k}, \eta_{jl}$  in (10) independent random variables with distribution  $N(0; 1)$ .

## 4 Simulation with given relative accuracy and reliability in $C([0, T])$

Let us study the rate of convergence in  $C([0, T])$  of model (10) to a process  $Y(t)$ . We will need several auxiliary facts.

**Lemma 4.1.** ([8]) Let  $X = \{X(t), t \in \mathbb{R}\}$  be a centered stochastic process which satisfies the requirements of Theorem 2.1,  $T > 0$ ,  $\phi$  be a scaling function,  $\psi$  be the corresponding wavelet, the function  $\hat{\phi}(y)$  be absolutely continuous on any interval, the function  $u(t, y)$  be absolutely continuous with respect to  $y$  for any fixed  $t$ , there exist the derivatives  $u'_\lambda(t, \lambda), \hat{\phi}'(y), \hat{\psi}'(y)$  and  $|\hat{\psi}'(y)| \leq C$ ,  $|u(t, \lambda)| \leq |t|u_1(\lambda)$ ,  $|u'_\lambda(t, \lambda)| \leq |t|u_2(\lambda)$ ,

$$\int_{\mathbb{R}} u_1(y)|y|dy < \infty, \quad \int_{\mathbb{R}} u_1(y)dy < \infty, \quad \int_{\mathbb{R}} u_1(y)|\hat{\phi}'(y)|dy < \infty, \quad (11)$$

$$\int_{\mathbb{R}} u_1(y)|\hat{\phi}(y)|dy < \infty, \quad \int_{\mathbb{R}} u_2(y)|y|dy < \infty, \quad \int_{\mathbb{R}} u_2(y)|\hat{\phi}(y)|dy < \infty, \quad (12)$$

$$\lim_{|y| \rightarrow \infty} u(t, y) \overline{\hat{\psi}(y/2^j)} = 0 \quad \forall j = 0, 1, \dots \quad \forall t \in [0, T]$$

and

$$\begin{aligned} \lim_{|y| \rightarrow \infty} u(t, y) \hat{\phi}(y) &= 0 \quad \forall t \in [0, T], \\ E_1 &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u_1(y) |\hat{\phi}(y)| dy, \\ E_2 &= \frac{1}{\sqrt{2\pi}} \left( \int_{\mathbb{R}} u_1(y) |\hat{\phi}'(y)| dy + \int_{\mathbb{R}} u_2(y) |\hat{\phi}(y)| dy \right), \\ F_1 &= \frac{C}{\sqrt{2\pi}} \int_{\mathbb{R}} u_1(y) |y| dy, \\ F_2 &= \frac{C}{\sqrt{2\pi}} \int_{\mathbb{R}} (u_1(y) + |y| u_2(y)) dy. \end{aligned}$$

Let the process  $\hat{X}(t)$  be defined by (8),  $\delta > 0$ . If  $N_0, N, M_j$  ( $j = 0, 1, \dots, N-1$ ) satisfy the inequalities

$$\begin{aligned} N_0 &> \frac{6}{\delta} E_2^2 T^2 + 1, \\ N &> \max \left\{ 1 + \log_2 \left( \frac{72 F_2^2 T^2}{5\delta} \right), 1 + \log_8 \left( \frac{18 F_1^2 T^2}{7\delta} \right) \right\}, \\ M_j &> 1 + \frac{12}{\delta} F_2^2 T^2, \end{aligned}$$

then

$$\sup_{t \in [0, T]} \mathbb{E} |X(t) - \hat{X}(t)|^2 \leq \delta. \quad (13)$$

**Lemma 4.2.** ([8]) Let  $X = \{X(t), t \in \mathbb{R}\}$  be a centered stochastic process which satisfies the requirements of Theorem 2.1,  $T > 0$ ,  $\phi$  be a scaling function,  $\psi$  be the corresponding wavelet,  $S(y) = \overline{\hat{\psi}(y)}$ ,  $S_\phi(y) = \overline{\hat{\phi}(y)}$ ;  $\phi(y), u(t, \lambda), S(y), S_\phi(y)$  satisfy such conditions: the function  $u(t, y)$  is absolutely continuous with respect to  $y$ , the function  $\hat{\phi}(y)$  is absolutely continuous,

$$|S'(y)| \leq M < \infty,$$

$$\lim_{|y| \rightarrow \infty} u(t, y) S(y/2^j) = 0, \quad j = 0, 1, \dots, \quad t \in [0, T],$$

$$\lim_{|y| \rightarrow \infty} u(t, y) S_\phi(y) = 0, \quad t \in [0, T],$$



there exist functions  $v(y)$  and  $w(y)$  such that

$$|u'_y(t_1, y) - u'_y(t_2, y)| \leq |t_2 - t_1|v(y),$$

$$|u(t_1, y) - u(t_2, y)| \leq |t_2 - t_1|w(y)$$

and

$$\begin{aligned} \int_{\mathbb{R}} |y|v(y)dy &< \infty, & \int_{\mathbb{R}} v(y)|S_\phi(y)|dy &< \infty, \\ \int_{\mathbb{R}} w(y)|S'_\phi(y)|dy &< \infty, & \int_{\mathbb{R}} w(y)dy &< \infty, \\ \int_{\mathbb{R}} w(y)|y|dy &< \infty, & \int_{\mathbb{R}} w(y)|S_\phi(y)|dy &< \infty; \end{aligned}$$

$a_{0k}(t)$  and  $b_{jl}(t)$  are defined by equalities (6) and (7),

$$A^{(1)} = \frac{1}{\sqrt{2\pi}} \left( \int_{\mathbb{R}} v(y)|S_\phi(y)|dy + \int_{\mathbb{R}} w(y)|S'_\phi(y)|dy \right),$$

$$B^{(0)} = \frac{M}{\sqrt{2\pi}} \int_{\mathbb{R}} w(y)|y|dy,$$

$$B^{(1)} = \frac{M}{\sqrt{2\pi}} \int_{\mathbb{R}} (w(y) + |y|v(y))dy,$$

$$C_{\Delta X} = \sqrt{\frac{2(A^{(1)})^2}{N_0 - 1} + \frac{(B^{(0)})^2}{7 \cdot 8^{N-1}} + \frac{(B^{(1)})^2}{2^{N-3}} + (B^{(1)})^2 \sum_{j=0}^{N-1} \frac{1}{2^{j-1}(M_j - 1)}}}.$$

Then for  $t_1, t_2 \in [0, T]$  and  $N > 1, N_0 > 1, M_j > 1$  the inequality

$$\begin{aligned} &\sum_{|k| \geq N_0} |a_{0k}(t_1) - a_{0k}(t_2)|^2 + \sum_{j \geq N} \sum_{l \in \mathbb{Z}} |b_{jl}(t_1) - b_{jl}(t_2)|^2 \\ &+ \sum_{j=0}^{N-1} \sum_{|l| \geq M_j} |b_{jl}(t_1) - b_{jl}(t_2)|^2 \leq C_{\Delta X}^2 (t_2 - t_1)^2 \end{aligned} \quad (14)$$

holds.

**Lemma 4.3.** *If*

$$N_0 \geq 1 + \frac{8(A^{(1)})^2}{\varepsilon^2},$$

$$N \geq \max \left\{ 1 + \log_8 \frac{4(B^{(0)})^2}{7\varepsilon^2}, 3 + \log_2 \frac{4(B^{(1)})^2}{\varepsilon^2} \right\},$$

$$M_j \geq 1 + 16 \frac{(B^{(1)})^2}{\varepsilon^2}$$

then

$$C_{\Delta X} \leq \varepsilon,$$

where  $A^{(1)}, B^{(0)}, B^{(1)}, C_{\Delta X}$  are defined in Lemma 4.2.

We omit the proof due to its triviality.

**Definition.** We say that a model  $\hat{Y}(t)$  approximates a stochastic process  $Y(t)$  with given *relative accuracy*  $\delta$  and *reliability*  $1 - \varepsilon$  (where  $\varepsilon \in (0; 1)$ ) in  $C([0, T])$  if

$$\mathbb{P} \left\{ \sup_{t \in [0, T]} |Y(t)/\hat{Y}(t) - 1| > \delta \right\} \leq \varepsilon.$$

Now we can formulate a result on the rate of convergence in  $C([0, T])$ .

**Theorem 4.1.** Suppose that a random process  $Y = \{Y(t), t \in \mathbb{R}\}$  can be represented as  $Y(t) = \exp\{X(t)\}$ , where a separable strictly sub-Gaussian random process  $X = \{X(t), t \in \mathbb{R}\}$  is mean square continuous, satisfies the condition RC and the conditions of Lemmas 4.1 and 4.2 together with a scaling function  $\phi$  and the corresponding wavelet  $\psi$ , the random variables  $\xi_{0k}, \eta_{jl}$  in expansion (5) of the process  $X(t)$  are independent strictly sub-Gaussian,  $\hat{X}(t)$  is a model of  $X(t)$  defined by (8),  $\hat{Y}(t)$  is defined by (10),  $\theta \in (0; 1)$ ,  $\delta > 0$ ,  $\varepsilon \in (0; 1)$ ,  $T > 0$ , the numbers  $A^{(1)}, B^{(0)}, B^{(1)}, E_2, F_1, F_2$  are defined in Lemmas 4.1 and 4.2,

$$\hat{\varepsilon} = \delta \sqrt{\varepsilon},$$

$$A(\theta) = \int_{1/(2\theta)}^{\infty} \frac{\sqrt{v+1}}{v^2} dv,$$

$$\tau_1 = \frac{e^{1/2} \hat{\varepsilon}}{2^{7/4} (64 + \hat{\varepsilon}^2)^{1/4}},$$

$$\tau_2 = (32 \ln(1 + \hat{\varepsilon}^2/60))^{1/2},$$

$$\tau_3 = \sqrt{\ln(1 + \hat{\varepsilon}^3/8)} / \sqrt{2},$$

$$\begin{aligned}
\tau_* &= \min\{\tau_1, \tau_2, \tau_3\}, \\
Q &= \frac{e^{1/2} \hat{\varepsilon} \theta(1 - \theta)}{2^{9/4} A(\theta) T(1 + \hat{\varepsilon}^3/8)}, \\
N_0^* &= 1 + \frac{8(A^{(1)})^2}{Q^2}, \\
N^* &= \max \left\{ 1 + \log_8 \frac{4(B^{(0)})^2}{7Q^2}, 3 + \log_2 \frac{4(B^{(1)})^2}{Q^2} \right\}, \\
M^* &= 1 + 16 \frac{(B^{(1)})^2}{Q^2}, \\
N_0^{**} &= \frac{6}{\tau_*^2} E_2^2 T^2 + 1, \\
N^{**} &= \max \left\{ 1 + \log_2 \left( \frac{72 F_2^2 T^2}{5 \tau_*^2} \right), 1 + \log_8 \left( \frac{18 F_1^2 T^2}{7 \tau_*^2} \right) \right\}, \\
M^{**} &= 1 + \frac{12}{\tau_*^2} F_2^2 T^2.
\end{aligned}$$

Suppose also that

$$\sup_{t \in [0, T]} \mathbf{E}(X(t) - \hat{X}(t))^2 > 0. \quad (15)$$

If

$$N_0 > \max\{N_0^*, N_0^{**}\}, \quad (16)$$

$$N > \max\{N^*, N^{**}\}, \quad (17)$$

$$M_j > \max\{M^*, M^{**}\} \quad (j = 0, 1, \dots, N - 1), \quad (18)$$

then the model  $\hat{Y}(t)$  approximates the process  $Y(t)$  with given relative accuracy  $\delta$  and reliability  $1 - \varepsilon$  in  $C([0, T])$ .

*Proof.* Denote

$$\Delta X(t) = X(t) - \hat{X}(t),$$

$$U(t) = Y(t)/\hat{Y}(t) - 1 = \exp\{\Delta X(t)\} - 1,$$

$$\rho_U(t, s) = \|U(t) - U(s)\|_{L_2(\Omega)},$$

$$\tau_{\Delta X} = \sup_{t \in [0, T]} \tau(\Delta X(t)).$$

Let us note that  $\rho_U$  is a pseudometric. Let  $N(u)$  be the metric massiveness of  $[0, T]$  with respect to  $\rho_U$ , i.e. the minimum number of closed balls in the space  $([0, T], \rho_U)$  with diameters at most  $2u$  needed to cover  $[0, T]$ ,

$$\varepsilon_0 = \sup_{t, s \in [0, T]} \rho_U(t, s).$$

We will denote the norm in  $L_2(\Omega)$  as  $\|\cdot\|_2$  below.

Since  $U(t) \in L_2(\Omega), t \in [0, T]$ , we obtain using Theorem 3.3.3 from [5] (see p. 98)

$$\mathbf{P} \left\{ \sup_{t \in [0, T]} |U(t)| > \delta \right\} \leq \frac{S_2^2}{\delta^2}, \quad (19)$$

where

$$S_2 = \sup_{t \in [0, T]} (\mathbf{E}|U(t)|^2)^{1/2} + \frac{1}{\theta(1-\theta)} \int_0^{\theta\varepsilon_0} N^{1/2}(u) du.$$

We will prove that  $S_2 \leq \delta\sqrt{\varepsilon} = \hat{\varepsilon}$ .

First of all let us estimate  $\mathbf{E}|U(t)|^2$ , where  $t \in [0, T]$ .

Using the inequality

$$|e^a - e^b| \leq |a - b| \max\{e^a, e^b\} \leq |a - b|(e^a + e^b) \quad (20)$$

(we set  $b = 0$ ) and Cauchy-Schwarz inequality we obtain

$$\mathbf{E}|U(t)|^2 = \mathbf{E}(\exp\{\Delta X(t)\} - 1)^2 \leq (\mathbf{E}|\Delta X(t)|^4)^{1/2} (\mathbf{E}(\exp\{\Delta X(t)\} + 1)^4)^{1/2}.$$

It follows from (4) that

$$\mathbf{E}|\Delta X(t)|^4 \leq \frac{32}{e^2} \tau_{\Delta X}^4. \quad (21)$$

Let us estimate  $G = \mathbf{E}(\exp\{\Delta X(t)\} + 1)^4$ . Since

$$\mathbf{E} \exp\{k\Delta X(t)\} \leq \exp\{k^2 \tau^2(\Delta X(t))/2\} = A^{k^2} \leq A^{16}, \quad 1 \leq k \leq 4,$$

where  $A = \exp\{\tau_{\Delta X}^2/2\}$ , we have

$$G \leq \sum_{k=1}^4 \binom{4}{k} A^{16} + 1 = 15A^{16} + 1. \quad (22)$$

It follows from Lemma 4.1 and (16)–(18) that

$$\tau_{\Delta X} = \sup_{t \in [0, T]} \mathbf{E}(|\Delta X(t)|^2)^{1/2} \leq \tau_*. \quad (23)$$

Using (21)–(23) we obtain

$$(\mathbf{E}|U(t)|^2)^{1/2} \leq \hat{\varepsilon}/2. \quad (24)$$

Let us estimate now

$$I(\theta) = \frac{1}{\theta(1-\theta)} \int_0^{\theta \varepsilon_0} N^{1/2}(u) du.$$

At first we will find an upper bound for  $N(u)$ . In order to do this we will prove that

$$\|U(t_1) - U(t_2)\|_2 \leq C_U |t_1 - t_2|, \quad (25)$$

where

$$C_U = (2^{9/4}/e^{1/2})C_{\Delta X} \exp\{2\tau_{\Delta X}^2\},$$

$C_{\Delta X}$  is defined in Lemma 4.2.

We have, using (20) and Cauchy-Schwarz inequality:

$$\begin{aligned} \|U(t_1) - U(t_2)\|_2^2 &= \mathbf{E}|\exp\{\Delta X(t_1)\} - \exp\{\Delta X(t_2)\}|^2 \\ &\leq \mathbf{E}|\Delta X(t_1) - \Delta X(t_2)|^2 (\exp\{\Delta X(t_1)\} + \exp\{\Delta X(t_2)\})^2 \\ &\leq (\mathbf{E}(\Delta X(t_1) - \Delta X(t_2))^4)^{1/2} (\mathbf{E}(\exp\{\Delta X(t_1)\} + \exp\{\Delta X(t_2)\})^4)^{1/2}. \end{aligned}$$

Applying (4), we obtain

$$(\mathbf{E}(\Delta X(t_1) - \Delta X(t_2))^4)^{1/2} \leq (2^{5/2}/e)C_{\Delta X}^2 |t_2 - t_1|^2. \quad (26)$$

Let us find an upper bound for

$$H = \mathbf{E}(\exp\{\Delta X(t_1)\} + \exp\{\Delta X(t_2)\})^4.$$

Since

$$\mathbf{E} \exp\{k\Delta X(t_1) + l\Delta X(t_2)\}$$

$$\begin{aligned} &\leq \exp\{\tau^2(k\Delta X(t_1) + l\Delta X(t_2))/2\} \leq \exp\{(k\tau(\Delta X(t_1)) + l\tau(\Delta X(t_2)))^2/2\} \\ &\leq \exp\{8\tau_{\Delta X}^2\}, \end{aligned}$$

where  $k + l = 4$ , we have:

$$H \leq \sum_{k=0}^4 \binom{4}{k} \exp\{8\tau_{\Delta X}^2\} = 16 \exp\{8\tau_{\Delta X}^2\} \quad (27)$$

and (25) follows from (26) and (27).

Using inequality (25), simple properties of metric entropy (see [5], Lemma 3.2.1, p. 88) and the inequality

$$N_{\rho_1}(u) \leq T/(2u) + 1$$

(where  $N_{\rho_1}$  is the entropy of  $[0, T]$  with respect to the Euclidean metric) we have

$$N(u) \leq \frac{TC_U}{2u} + 1.$$

Since  $\varepsilon_0 \leq C_U T$  we obtain

$$\begin{aligned} &\int_0^{\theta\varepsilon_0} N^{1/2}(u) du \leq \int_0^{\theta\varepsilon_0} (TC_U/(2u) + 1)^{1/2} du \\ &= \frac{TC_U}{2} \int_{TC_U/(2\theta\varepsilon_0)}^{\infty} \frac{\sqrt{v+1}}{v^2} dv \leq TC_U A(\theta)/2. \end{aligned} \quad (28)$$

It is easy to check using Lemma 4.3 that under the conditions of the theorem the inequality

$$C_{\Delta X} \leq Q \quad (29)$$

holds. It follows from (23) and (29) that

$$C_U \leq \frac{\hat{\varepsilon} \theta(1 - \theta)}{TA(\theta)}$$

and therefore using (28) we obtain

$$I(\theta) \leq \hat{\varepsilon}/2. \quad (30)$$

Now the statement of the theorem follows from (19), (24) and (30).  $\square$

**Example 4.1.** Let us consider a function  $u(t, \lambda) = t/(1 + t^2 + \lambda^2)^4$  and an arbitrary Daubechies wavelet (with the corresponding scaling function  $\phi$  and the wavelet  $\psi$ ). We will use the notations

$$a_{0k}(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(t, y) \overline{\hat{\phi}_{0k}(y)} dy, \quad b_{jl}(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(t, y) \overline{\hat{\psi}_{jl}(y)} dy$$

and consider the stochastic process

$$X(t) = \sum_{k \in \mathbb{Z}} \xi_{0k} a_{0k}(t) + \sum_{j=0}^{\infty} \sum_{l \in \mathbb{Z}} \eta_{jl} b_{jl}(t),$$

where  $\xi_{0k}, \eta_{jl}$  ( $k, l \in \mathbb{Z}, j = 0, 1, \dots$ ) are independent uniformly distributed over  $[-\sqrt{3}, \sqrt{3}]$ . It is easy to see that the process  $Y(t) = \exp\{X(t)\}$  and the Daubechies wavelet satisfy the conditions of Theorem 4.1.

## 5 Simulation with given accuracy and reliability in $L_p([0, T])$

Now we will consider the rate of convergence in  $L_p([0, T])$  of model (10) to a process  $Y(t)$ .

**Lemma 5.1.** *Suppose that a centered stochastic process  $X = \{X(t), t \in \mathbb{R}\}$  satisfies the conditions of Theorem 2.1,  $\phi$  is a scaling function,  $\psi$  is the corresponding wavelet,  $\hat{\phi}$  and  $\hat{\psi}$  are Fourier transforms of  $\phi$  and  $\psi$  respectively,  $\hat{\phi}(y)$  is absolutely continuous,  $u(t, y)$  is defined in Theorem 2.1 and  $u(t, y)$  is absolutely continuous for any fixed  $t$ , there exist derivatives  $u'_y(t, y), \hat{\phi}'(y), \hat{\psi}'(y)$  and  $|\hat{\psi}'(y)| \leq C, |u(t, y)| \leq u_1(y), |u'_y(t, y)| \leq |t| u_2(y)$ , equalities (11) and (12) hold,*

$$\lim_{|y| \rightarrow \infty} u(t, y) \overline{\hat{\psi}(y/2^j)} = 0 \quad \forall j = 0, 1, \dots \quad \forall t \in \mathbb{R},$$

$$\lim_{|y| \rightarrow \infty} u(t, y) |\hat{\phi}(y)| = 0 \quad \forall t \in \mathbb{R};$$

$$S_1 = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u_1(y) |\hat{\phi}'(y)| dy, \quad S_2 = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u_2(y) |\hat{\phi}(y)| dy,$$

$$Q_1 = \frac{C}{\sqrt{2\pi}} \int_{\mathbb{R}} u_1(y) dy, \quad Q_2 = \frac{C}{\sqrt{2\pi}} \int_{\mathbb{R}} u_2(y) |y| dy.$$

Then the following inequalities hold for the coefficients  $a_{0k}(t), b_{jl}(t)$  in expansion (5) of the process  $X(t)$ :

$$|a_{00}(t)| \leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u_1(y) |\hat{\phi}(y)| dy, \quad (31)$$

$$|b_{j0}(t)| \leq \frac{C}{\sqrt{2\pi} 2^{3j/2}} \int_{\mathbb{R}} u_1(y) |y| dy, \quad j = 0, 1, \dots, \quad (32)$$

$$|a_{0k}(t)| \leq \frac{S_1 + S_2 |t|}{|k|}, \quad k \neq 0, \quad (33)$$

$$|b_{jl}(t)| \leq \frac{Q_1 + Q_2 |t|}{2^{j/2} |k|}, \quad k \neq 0, \quad j = 0, 1, \dots \quad (34)$$

The proof of inequalities (31)–(34) is analogous to the proof of similar inequalities for the coefficients of expansion (5) of a stationary process in [7].

**Lemma 5.2.** *Suppose that a random process  $X = \{X(t), t \in \mathbb{R}\}$  satisfies the conditions of Theorem 2.1; a scaling function  $\phi$  and the corresponding wavelet  $\psi$  together with the process  $X(t)$  satisfy the conditions of Lemma 5.1,  $C, Q_1, Q_2, S_1, S_2, u_1(y)$  are defined in Lemma 5.1,  $T > 0, p \geq 1, \delta \in (0; 1), \varepsilon > 0$ ,*

$$\delta_1 = \min \left\{ \frac{\varepsilon^2}{2T^{2/p} \ln(2/\delta)}, \frac{\varepsilon^2}{pT^{2/p}} \right\}, \quad D = \frac{C}{\sqrt{2\pi}} \int_{\mathbb{R}} u_1(y) |y| dy.$$

If

$$N_0 > \frac{6}{\delta_1} (S_1 + S_2 T)^2 + 1,$$

$$N > \max \left\{ 1 + \log_2 \left( \frac{72(Q_1 + Q_2 T)^2}{5\delta_1} \right), 1 + \log_8 \left( \frac{18D^2}{7\delta_1} \right) \right\},$$

$$M_j > 1 + \frac{12}{\delta_1} (Q_1 + Q_2 T)^2 \left( 1 - \frac{1}{2^N} \right),$$

then

$$\sup_{t \in [0, T]} \mathbb{E} |X(t) - \hat{X}(t)|^2 \leq \delta_1.$$



*Proof.* We have

$$\mathbb{E}|X(t) - \hat{X}(t)|^2 = \sum_{k:|k| \geq N_0} |a_{0k}(t)|^2 + \sum_{j=0}^{N-1} \sum_{l:|l| \geq M_j} |b_{jl}(t)|^2 + \sum_{j=N}^{\infty} \sum_{l \in \mathbb{Z}} |b_{jl}(t)|^2.$$

It remains to apply inequalities (31)–(34).  $\square$

**Definition.** We say that a model  $\hat{Y}(t)$  approximates a stochastic process  $Y(t)$  with given *accuracy*  $\delta$  and *reliability*  $1 - \varepsilon$  (where  $\varepsilon \in (0; 1)$ ) in  $L_p([0, T])$  if

$$\mathbb{P} \left\{ \left( \int_0^T |Y(t) - \hat{Y}(t)|^p dt \right)^{1/p} > \delta \right\} \leq \varepsilon.$$

**Theorem 5.1.** Suppose that a random process  $Y = \{Y(t), t \in \mathbb{R}\}$  can be represented as  $Y(t) = \exp\{X(t)\}$ , where a separable strictly sub-Gaussian random process  $X = \{X(t), t \in \mathbb{R}\}$  is mean square continuous, satisfies the condition RC and the conditions of Lemma 5.2 together with a scaling function  $\phi$  and the corresponding wavelet  $\psi$ , the random variables  $\xi_{0k}, \eta_{jl}$  in expansion (5) of the process  $X(t)$  are independent strictly sub-Gaussian,  $\hat{X}(t)$  is a model of  $X(t)$  defined by (8),  $\hat{Y}(t)$  is defined by (10),  $D, Q_1, Q_2, S_1, S_2$  are defined in Lemmas 5.1 and 5.2,  $\delta > 0$ ,  $\varepsilon \in (0; 1)$ ,  $p \geq 1$ ,  $T > 0$ .

Let

$$m = \frac{\varepsilon \delta^p}{2^{2p}(p/e)^{p/2} T \sup_{t \in [0, T]} (\mathbb{E} \exp\{2pX(t)\})^{1/2}},$$

$$h(t) = t^p(1 + \exp\{8p^2 t^2\})^{1/4}, \quad t \geq 0,$$

$x_m$  be the root of the equation

$$h(x) = m.$$

If

$$N_0 > \frac{6}{x_m^2} (S_1 + S_2 T)^2 + 1, \quad (35)$$

$$N > \max \left\{ 1 + \log_2 \left( \frac{72(Q_1 + Q_2 T)^2}{5x_m^2} \right), 1 + \log_8 \left( \frac{18D^2}{7x_m^2} \right) \right\}, \quad (36)$$

$$M_j > 1 + \frac{12}{x_m^2}(Q_1 + Q_2 T)^2 \left(1 - \frac{1}{2^N}\right) \quad (j = 0, 1, \dots, N-1), \quad (37)$$

then the model  $\hat{Y}(t)$  defined by (10) approximates  $Y(t)$  with given accuracy  $\delta$  and reliability  $1 - \varepsilon$  in  $L_p([0, T])$ .

*Proof.* We will use the following notations:

$$\Delta X(t) = \hat{X}(t) - X(t),$$

$$\bar{\tau}_X = \sup_{t \in [0, T]} \tau(X(t)),$$

$$\bar{\tau}_{\Delta X} = \sup_{t \in [0, T]} \tau(\Delta X(t)),$$

$$c_p = 2(4p/e)^{2p}.$$

We will denote the norm in  $L_p([0, T])$  as  $\|\cdot\|_p$ .

Let us estimate  $P\{\|Y - \hat{Y}\|_p > \delta\}$ . We have

$$\begin{aligned} P\{\|Y - \hat{Y}\|_p > \delta\} &\leq \frac{\mathbb{E}\|Y - \hat{Y}\|_p^p}{\delta^p} \\ &= \frac{\mathbb{E} \int_0^T |\exp\{X(t)\} - \exp\{\hat{X}(t)\}|^p dt}{\delta^p}. \end{aligned} \quad (38)$$

Denote

$$\Delta(t) = \mathbb{E}|\exp\{X(t)\} - \exp\{\hat{X}(t)\}|^p.$$

An application of Cauchy-Schwarz inequality yields:

$$\begin{aligned} \Delta(t) &= \mathbb{E} \exp\{pX(t)\} |1 - \exp\{\Delta X(t)\}|^p \\ &\leq (\mathbb{E} \exp\{2pX(t)\})^{1/2} (\mathbb{E} |1 - \exp\{\Delta X(t)\}|^{2p})^{1/2}. \end{aligned} \quad (39)$$

We will need two auxiliary inequalities. Using the power mean inequality

$$\frac{a+b}{2} \leq \left(\frac{a^r + b^r}{2}\right)^{1/r},$$

where  $r \geq 1$ , and setting  $a = e^c$ ,  $b = 1$  we obtain

$$(e^c + 1)^r \leq 2^{r-1}(e^{cr} + 1). \quad (40)$$

It follows from (20) that

$$|e^a - 1|^q \leq |a|^q (e^a + 1)^q \quad (41)$$

for  $q \geq 0$ .

Now let us estimate  $\mathbb{E}|1 - \exp\{\Delta X(t)\}|^{2p}$ , where  $t \in [0, T]$ , using (41):

$$\begin{aligned} \mathbb{E}|1 - \exp\{\Delta X(t)\}|^{2p} &\leq \mathbb{E}|\Delta X(t)|^{2p} (1 + \exp\{\Delta X(t)\})^{2p} \\ &\leq (\mathbb{E}|\Delta X(t)|^{4p})^{1/2} (\mathbb{E}(1 + \exp\{\Delta X(t)\})^{4p})^{1/2}. \end{aligned} \quad (42)$$

Applying (40) we obtain:

$$\mathbb{E}(1 + \exp\{\Delta X(t)\})^{4p} \leq 2^{4p-1} \mathbb{E}(\exp\{4p\Delta X(t)\} + 1). \quad (43)$$

It follows from (39), (42) and (43) that for  $t \in [0, T]$

$$\Delta(t) \leq 2^{p-1/4} (\mathbb{E} \exp\{2pX(t)\})^{1/2} (\mathbb{E}|\Delta X(t)|^{4p})^{1/4} (1 + \mathbb{E} \exp\{4p\Delta X(t)\})^{1/4}. \quad (44)$$

Since for  $t \in [0, T]$

$$\mathbb{E}|\Delta X(t)|^{4p} \leq c_p \bar{\tau}_{\Delta X}^{4p}$$

(see (4)) and

$$\mathbb{E} \exp\{4p\Delta X(t)\} \leq \exp\{8p^2 \bar{\tau}_{\Delta X}^2\}$$

(see (3)) we have

$$\Delta(t) \leq 2^{p-1/4} c_p^{1/4} \sup_{t \in [0, T]} (\mathbb{E} \exp\{2pX(t)\})^{1/2} h(\bar{\tau}_{\Delta X}), \quad t \in [0, T]. \quad (45)$$

It follows from Lemma 5.2 and inequalities (35)–(37) that

$$\bar{\tau}_{\Delta X} = \sup_{t \in [0, T]} (\mathbb{E}(X(t) - \hat{X}(t))^2)^{1/2} \leq x_m.$$

We obtain using (45) that

$$\Delta(t) \leq \varepsilon \delta^p / T, \quad t \in [0, T],$$

and hence

$$\mathbb{E}\|Y - \hat{Y}\|_p^p = \int_0^T \Delta(t) dt \leq \varepsilon \delta^p. \quad (46)$$

Now the statement of the theorem follows from (38) and (46).  $\square$

**Example 5.1.** Let us consider a centered Gaussian process  $X(t)$  with the correlation function

$$R(t, s) = \int_{\mathbb{R}} u(t, y) u(s, y) dy,$$

where

$$u(t, y) = \frac{t}{1 + t^2 + \exp\{y^2\}},$$

and an arbitrary Battle-Lemarié wavelet. It is easy to check that the process  $Y(t) = \exp\{X(t)\}$  and the Battle-Lemarié wavelet satisfy the conditions of Theorem 5.1.

## Acknowledgments

The author's research was supported by a Swiss Government Excellence Scholarship. The author would like to thank professors Enkelejd Hashorva and Yuriy V. Kozachenko for valuable discussions.

## References

- [1] Ogorodnikov, V.A, Prigarin, S.M., 1996. *Numerical Modelling of Random Processes and Fields: Algorithms and Applications*. VSP, Utrecht.
- [2] Ripley, B.D., 1987. *Stochastic Simulation*. John Wiley & Sons, New York.

- [3] Kozachenko, Yu., Pogoriliak, O., 2011. Simulation of Cox processes driven by random Gaussian field. *Methodol. Comput. Appl. Probab.* 13: 511–521.
- [4] Kozachenko, Yu., Sottinen, T., Vasylyk, O., 2005. Simulation of weakly self-similar stationary increment  $\text{Sub}_\varphi(\Omega)$ -processes: a series expansion approach. *Methodol. Comput. Appl. Probab.* 7: 379–400.
- [5] Buldygin, V.V, Kozachenko, Yu.V., 2000. *Metric Characterization of Random Variables and Random Processes*. Amer. Math. Soc., Providence, RI.
- [6] Hernández, E., Weiss, G., 1996. *A First Course on Wavelets*. CRC press, Boca Raton.
- [7] Kozachenko, Yu., Turchyn, Ye., 2008. On Karhunen-Loeve-like expansion for a class of random processes. *Internat. J. of Statist. and Management System* 3: 43–55.
- [8] Turchyn, Y., 2011. Simulation of sub-Gaussian processes using wavelets. *Monte Carlo Methods Appl.* 17: 215–231.